DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

350. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

Solve the equations:
$$x+y+z=a_0,$$
$$x+yu+zv=a_1,$$
$$x+yu^2+zv^2=a_2,$$
$$x+yu^3+zv^3=a_3,$$
$$x+yu^4+zv^4=a_4.$$

Solution by A. H. HOLMES, Brunswick, Maine, and the PROPOSER.

$x+y+z=a_0$...(1); $x+yu+zu=a_1$...(2); $x+yu^2+zv^2=a_2$...(3); $x+yu^3+zu^3=a_2$...(4); and $x+yu^4+zu^4=a_4$...(5).

Subtracting (1) from (2), (2) from (3), (3) from (4), and (4) from (5), we have,

$$y(u-1)+z(v-1)=a_1-a_0\text{...(6)};$$
$$yu(u-1)+zu(v-1)=a_2-a_1\text{...(7)};$$
$$yu^2(u-1)+zv^2(v-1)=a_3-a_2\text{...(8); and}$$
$$yu^3(u-1)+zv^3(v-1)=a_4-a_3\text{...(9)}.$$

Eliminating y from (6), (7), (8), and (9), we have,

$$(a_1-a_0)u-zu(v-1)=a_2-a_1-zv(v-1)\text{...(10)},$$
$$(a_2-a_1)u-zuv(v-1)=a_3-a_2-zv^2(v-1)\text{...(11), and}$$
$$(a_3-a_2)u-zuv^2(v-1)=a_4-a_3-zv^3(v-1)\text{...(12)}.$$

Eliminating z from (10), (11), and (12), we have,

$$(a_1-a_0)uv-(a_2-a_1)v=(a_2-a_1)u-(a_3-a_2)\text{...(13), and}$$
$$(a_2-a_1)uv-(a_3-a_2)v=(a_3-a_2)u-(a_4-a_3)\text{...(14)}.$$

Eliminating v from (13) and (14), we have

$$\frac{(a_3-a_2)u-(a_4-a_3)}{(a_2-a_1)u-(a_3-a_2)}=\frac{(a_2-a_1)u-(a_3-a_2)}{(a_1-a_0)u-(a_2-a_1)}.$$

Putting $a_1-a_0=d_1$, $a_2-a_1=d_2$, $a_3-a_2=d_3$, $a_4-a_3=d_4$, and solving,

$$u=\frac{d_1d_4-d_2d_3\pm\sqrt{[(d_1d_4-d_2d_3)^2+4(d_1d_3-d_2^2)(d_2^2-d_2d_4)]}}{2(d_1d_3-d_2^2)}.$$

From (13) we find v. Then from (10) we find z. From (6), y, and from (1), x.

351. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve, $y^2+yz+z^2=a^2\ldots(1)$.
$z^2+zx+x^2=b^2\ldots(2)$.
$x^2+xy+y^2=c^2\ldots(3)$.

I. Solution by J. A. COLSON, Searsport, Maine.

$b^2-c^2=(z-y)(x+y+z)$. $\therefore(b^2-c^2)x=(zx-xy)(x+y+z)$.

$c^2-a^2=(x-z)(x+y+z)$. $\therefore(c^2-a^2)y=(xy-yz)(x+y+z)$.

$a^2-b^2=(y-x)(x+y+z)$. $\therefore(a^2-b^2)z=(yz-zx)(x+y+z)$.

$\therefore(b^2-c^2)x+(c^2-a^2)y+(a^2-b^2)z=0$.

For convenience, put $b^2-c^2=f$, $c^2-a^2=g$, and $a^2-b^2=h$. Then $f+g+h=0$, and $fx+gy+hz=0$.

$$\therefore z=-\frac{fx+gy}{h},\text{ and } x+y+z=x+y-\frac{fx+gy}{h}=\frac{(h-f)x+(h-g)y}{h}.$$

$$\therefore a^2-b^2=h=(y-x)(x+y+z)=(y-x)\frac{(h-f)x+(h-g)y}{h}.$$

$\therefore h^2=(f-h)x^2+(g-f)xy+(h-g)y^2$.

But from (3) we have $y^2=c^2-x^2-xy$.

Hence, $h^2=c^2(h-g)+(f+g-2h)x^2+(2g-f-h)xy=c^2(h-g)-3hx^2+3gxy$.

$$\therefore y=\frac{3hx^2+h^2+c^2(g-h)}{3gx}.$$

Substitute in (3), and we have

$$x^2+\frac{3hx^2+h^2+c^2(g-h)}{3g}+\frac{[3hx^2+h^2+c^2(g-h)]^2}{9g^2x^2}-c^2=0.$$

Hence, clearing of fractions and uniting, we have,

$$9(g^2+gh+h^2)x^4-3[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2$$
$$+[h^2+c^2(g-h)]^2=0.$$

$$\therefore 36(g^2+gh+h^2)^2x^4-12(g^2+gh+h^2)[c^2(2g^2-gh+2h^2)-h^2(g+2h)]x^2$$
$$+[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2=[c^2(2g^2-gh+2h^2)-h^2(g+2h)]^2-4(g^2$$
$$+gh+h^2)[h^2-c^2(g-h)]^2=9c^4g^2h^2-6c^2g^2h^2(2g+h)-3g^2h^4.$$

$$\therefore 6(g^2+gh+h^2)x^2-[c^2(2g^2-gh+2h^2)-h^2(g+2h)]$$
$$=\pm gh\sqrt{[9c^4-6c^2(2g+h)-3h^2]}.$$